PHYSICAL REVIEW E **79**, 046601 (2009)

Kink motion in a curved Josephson junction

T. Dobrowolski
Institute of Physics AP, Podchorążych 2, 30-084 Cracow, Poland
(Received 13 November 2008; revised manuscript received 2 February 2009; published 2 April 2009)

The propagation of the kink along the curved Josephson junction is considered. It is shown that the torsion as well as curvature is responsible for the existence of the energy barrier that affects the motion of the fluxon in the junction. It is also observed that the motion of the kink in the junction can be controlled in a very simple way even in the straight junction by modulating its width.

DOI: 10.1103/PhysRevE.79.046601 PACS number(s): 05.45.Yv, 85.25.Cp

I. INTRODUCTION

The Josephson effect was first predicted by Josephson [1] and then observed experimentally by Anderson and Rowell [2]. The effect occurs in two superconductors separated by a very thin layer of insulator or normal metal, in which the current is carried across the tunneling barrier by Cooper pairs. Although the Josephson junction is a two-dimensional system, if the transverse dimension is smaller than the Josephson length, then the system can be considered as the one-dimensional (1-d) system called the long Josephson junction. The dynamics of the phase difference between the two superconducting electrodes ϕ in this system is described by the sine-Gordon model [3]. The model appears in description of many systems [4]. The solutions of this model in 1 +1 dimensions have been extensively studied for many years (see, for example, [5]). Moreover, investigations on solutions of this model in a higher number of dimensions also have a long history (see, for instance, [6]). The best-known example of the solution of this model is the kink solution interpolating between various ground states of the potential. In the theory of Josephson junction the kink represents a fluxon, i.e., a quantum of magnetic flux.

The recent progress in microtechnology made it possible to fabricate various low-dimensional systems with complicated geometry. In particular, one can produce the Josephson junction with an almost arbitrary shape. The question about the influence of the nontrivial geometry on the kink motion was first raised in [7]. The authors proposed an effective description of the long Josephson junction in a particular case when the junction is represented by a curve embedded in a plane. The aim of the present paper is to implement a formalism that makes it possible to describe the influence of the nontrivial geometry of the Josephson junction on the kink motion in the case of an arbitrarily deformed long Josephson junction and also in the case of a large-area Josephson junction. In the case of the long Josephson junction, which is a one-dimensional system, we describe the effects of its embedding [in a three-dimensional (3-d) space] on the kink motion. Using the same formalism, we are also able to describe similar effects in the case of the large-area Josephson junction, which is a two-dimensional system. In two dimensions, the proposed formalism works well, if the junction is flat in one of the normal directions. In both cases, i.e., one and two dimensional, we focus our studies on considerations of the kink dynamics in the Josephson junction whose width is much smaller than its length. In the paper, we use normalized units (t,x^i). The space coordinates are normalized to the Josephson length λ_J,

$$x^i = X^i/\lambda_J,$$

and the time is normalized as follows:

$$t = \omega_P T,$$

where ω_P is plasma frequency. Here (T,X^i) are usual Cartesian coordinates. We also neglect the quasiparticle current and assume a zero external bias current and absence of the external magnetic field.

The paper is organized as follows. In Sec. II, we introduce certain curved coordinates on the basis of the junction and fix our notation. Section III is devoted to the construction of an effective Lagrangian. The effective Lagrangian defines the dynamics of the variable that describes the time dependence of the position of the kink. The main advantage of the proposed formalism is the possibility of describing the influence of an arbitrary deformation of the junction (i.e., the central line of the junction) on the kink motion. Section IV contains remarks.

II. PRELIMINARIES ON THE EMBEDDING OF THE JOSEPHSON LINE

We should like to consider the sine-Gordon model defined by the Lagrangian density

$$\mathcal{L} = \frac{1}{2}\eta_M^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi), \tag{1}$$

where $\eta_M^{\mu\nu}$ is Minkowski metric in Cartesian coordinates $x^\mu = (x^0,x^1,x^2,x^3) = (t,x,y,z)$, i.e., $\eta_M^{\mu\nu} = \mathrm{diag}(+1,-1,-1,-1)$; here we consider the potential $V(\phi) = 1-\cos\phi$. The equation of motion is the following:

$$\partial_t^2\phi - \Delta\phi + \sin\phi = 0. \tag{2}$$

In the case of the straight and long Josephson junction, equation of motion (2) possesses static solution

$$\phi_K(x,t) = 4\arctan(e^x), \tag{3}$$

which can be easily generalized to stationary kink solution. Our purpose in this paper is consideration of the influence of the curvature and the torsion of the junction on the kink

1539-3755/2009/79(4)/046601(7)

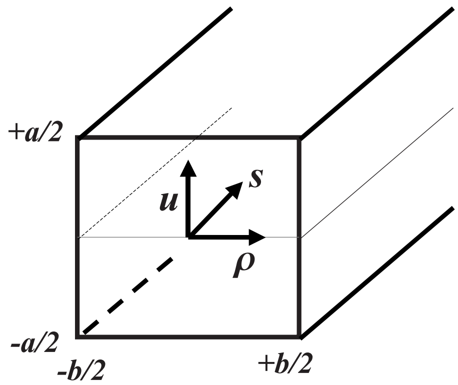

FIG. 1. The coordinate s parametrizes the points of the central line of the junction. The normal variable ρ indicates the positions of the points at the insulator plane, and u is a variable which indicates a position in the direction perpendicular to the insulator (or normal-metal) plane.

motion. In order to realize this purpose we separate the space and the time variables in the Lagrangian density of the model,

$$\mathcal{L} = \frac{1}{2}(\partial_t \phi)^2 - \frac{1}{2}\eta_E^{ij}\partial_i \partial_j \phi - V(\phi), \quad (4)$$

where η_E^{ij} is Euclidean metric diag$(+1,+1,+1)$ in Cartesian coordinates $(x^i)=(x^1,x^2,x^3)$.

In description of the curved Josephson junction we use suitable curved coordinates. One coordinate, denoted by s, is arc-length coordinate located in the central line of the junction, the second u is the transversal coordinate orthogonal to s, and the last coordinate ρ is orthogonal to s and u (see Fig. 1). The Lagrangian density in these coordinates is the following:

$$\mathcal{L} = \frac{1}{2}(\partial_t \phi)^2 - \frac{1}{2}G^{\alpha\beta}\partial_\alpha \partial_\beta \phi - V(\phi), \quad (5)$$

where we use the notation $\xi^\alpha=(\xi^1,\xi^2,\xi^3)=(\rho^1,\rho^2,s)=(u,\rho,s)$. We treat the junction as a one-dimensional object and therefore we can identify its points by the vector field $\vec{X}(s)$. At this point we would like to underline that our proposed formalism makes possible also the description of the two-dimensional junction in case of the junction that is flat in the direction of the normal variable ρ.

In the neighborhood of the Josephson line, we introduce the curved coordinates $(\xi^\alpha)=(\rho^i,s)$,

$$\vec{x} = \vec{X}(s) + \rho^j \vec{n}_j(s), \quad (6)$$

where the vectors $\vec{n}_j$ are normal to the line of the junction and ρ^j are coordinates in the directions of the vectors $\vec{n}_j$. The above formula establishes implicit connection between the Cartesian coordinates x^i and curved coordinates (ρ^i,s). By definition the vectors $\vec{n}_j$ are normalized to unity and are orthogonal each other and to the tangent vector $\vec{X}_{,s}=\partial_s\vec{X}$ to the central line of the junction,

$$\vec{n}_i\vec{n}_j = \delta_{ij}, \quad \vec{n}_j\vec{X}_{,s} = 0, \quad (7)$$

where the scalar products are calculated with respect to Euclidean metric $(+,+,+)$ in Cartesian coordinates. In the arc-length parametrization the tangent vector has unit length

$$\vec{X}_{,s}\vec{X}_{,s} = 1. \quad (8)$$

Three vectors $\vec{X}_{,s}$, $\vec{n}_1$, and $\vec{n}_2$ form the Frenet frame (see Fig. 1). The typical choice of the normals, i.e., the normal vector $\vec{n}_1$ and the binormal vector $\vec{n}_2$, is the following:

$$\vec{n}_1 = \frac{\vec{X}_{,ss}}{|\vec{X}_{,ss}|}, \quad \vec{n}_2 = \vec{X}_{,s} \times \vec{n}_1. \quad (9)$$

The embedding of the line in 3-d space is described by the extrinsic curvature K and the torsion coefficient ω and they follow from the Frenet-Serret formulas,

$$\partial_s \vec{X}_{,s} = \vec{X}_{,ss} = K\vec{n}_1, \quad (10)$$

$$\partial_s \vec{n}_1 = \vec{n}_{1,s} = -K\vec{X}_{,s} + \omega\vec{n}_2, \quad (11)$$

$$\partial_s \vec{n}_2 = \vec{n}_{2,s} = -\omega\vec{n}_1. \quad (12)$$

From these equations we can calculate the curvature and the torsion coefficients. From the square of the first Frenet-Serret equation (10), we obtain the curvature coefficient

$$K = |\vec{X}_{,ss}|. \quad (13)$$

If we project the third Eq. (12) onto normal direction $\vec{n}_1$ and then use the explicit form of the binormal vector $\vec{n}_2$ [Eq. (9)], we obtain the torsion coefficient

$$\omega = -\vec{n}_1\partial_s(\vec{X}_{,s} \times \vec{n}_1). \quad (14)$$

The normal vector $\vec{n}_1$ can be eliminated completely, from the last formula, if we use the first Frenet-Serret equation and explicit expression for curvature (13),

$$\omega = \frac{1}{\vec{X}_{,ss}^2}\vec{X}_{,s}(\vec{X}_{,ss} \times \vec{X}_{,sss}). \quad (15)$$

An example of the flat curve, i.e., the curve located in the plane, is a circle. In this case we have the following radius vector:

$$\vec{X}(s) = R\cos(s/R)\vec{e}_x + R\sin(s/R)\vec{e}_y + z_0\vec{e}_z, \quad (16)$$

where R is the radius of the circle and z_0 specifies the plane where the circle is located. The curvature and the torsion in this example are the following:

$$K = \frac{1}{R}, \quad \omega = 0. \quad (17)$$

The other example of the curve, which is not flat, is a helix. The geometry of this curve is richer in this sense that it possesses nonzero curvature and also nonzero torsion coefficient

$$\vec{X}(s) = R\cos(Cs)\vec{e}_x + R\sin(Cs)\vec{e}_y + hCs\vec{e}_z, \tag{18}$$

where R is the radius, h measures the slip of the helix, and $C=1/\sqrt{R^2+h^2}$. If we calculate the curvature and the torsion, then we obtain

$$K = \frac{R}{R^2+h^2}, \quad \omega = \frac{h}{R^2+h^2}. \tag{19}$$

In the calculus we shall need the metric in the curved coordinates $(\xi^\alpha)=(\rho^i,s)$,

$$G_{\alpha\beta} = \frac{\partial x^i}{\partial \xi^\alpha}\frac{\partial x^j}{\partial \xi^\beta}\eta^E_{ij}. \tag{20}$$

The components of this metric are the following:

$$G_{ij} = \delta_{ij}, \quad G_{is} = -\omega\varepsilon_{ij}\rho^j, \quad G_{ss} = (1-uK)^2 + \rho^i\rho^i\omega. \tag{21}$$

In our computations we use the Lagrangian density in curved coordinates and therefore we need the components of the inverse metric in curved coordinates:

$$G^{ij} = \delta^{ij} + \frac{\omega^2}{G}\varepsilon^{il}\varepsilon^{jk}\rho^l\rho^k, \quad G^{is} = \frac{\omega}{G}\varepsilon^{ij}\rho^j, \quad G^{ss} = \frac{1}{G}, \tag{22}$$

where

$$G = (1-uK)^2 \tag{23}$$

is the determinant of the space metric $G_{\alpha\beta}$ in curved coordinates.

III. EFFECTIVE DESCRIPTION OF THE KINK MOTION IN CURVED JOSEPHSON JUNCTION

First, we transform Lagrangian density (5) to the curved coordinates. If we use formulas (22), then the Lagrangian density can be converted to the form

$$\mathcal{L} = \frac{1}{2}(\partial_t\phi)^2 - \frac{1}{2}(\partial_i\phi)(\partial_i\phi) - \frac{\omega^2}{2G}\varepsilon^{il}\varepsilon^{jk}\rho^l\rho^k(\partial_i\phi)(\partial_j\phi) - \frac{\omega}{G}\varepsilon^{ij}\rho^j(\partial_i\phi)(\partial_s\phi) - \frac{1}{2G}(\partial_s\phi)^2 - V(\phi). \tag{24}$$

The effective Lagrangian which describes the motion of the kink along the junction can be obtained by integration of Lagrangian density (24) over the spatial variables,

$$L = \int_0^l ds \int_{-a/2}^{+a/2} du \int_{-b/2}^{+b/2} d\rho \sqrt{G}\mathcal{L}(\phi_K). \tag{25}$$

The effective Lagrangian in a natural way separates into two parts: kinetic and potential energies,

$$L = T - U. \tag{26}$$

The kinetic energy is defined by the integral of the first term of expression (24),

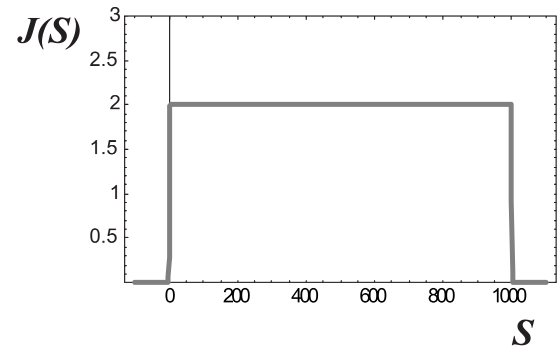

FIG. 2. The function $J(S)$, independent of the values of parameters, is equal to 2. The parameter l in the plot is equal to 1000.

$$T = \int_0^l ds \int_{-a/2}^{+a/2} du \int_{-b/2}^{+b/2} d\rho \sqrt{G}\frac{1}{2}(\partial_t\phi_K)^2, \tag{27}$$

and the effective potential energy comes from the rest of Lagrangian density (24). In this paper we consider two geometrically different situations.

A. Case 1

In the first case the system is generically one dimensional, i.e., due to the form of the kink profile $[\phi_K=\phi_K(t,s)]$, the junction is or can be treated as a one-dimensional object. In this case the kink is one- or quasi-one-dimensional solution $(\partial_i\phi_K=0)$ and the Lagrangian density simplifies to the form

$$\mathcal{L} = \frac{1}{2}(\partial_t\phi_K)^2 - \frac{1}{2G}(\partial_s\phi_K)^2 - V(\phi_K). \tag{28}$$

The effective potential in this simple case is given by the integral

$$U = \int_0^l ds \int_{-a/2}^{+a/2} du \int_{-b/2}^{+b/2} d\rho \sqrt{G}\left[\frac{1}{2G}(\partial_s\phi_K)^2 + (1-\cos\phi_K)\right]. \tag{29}$$

An example of the kink configuration of this type has the following analytical form:

$$\phi_K = 4\arctan[e^{s-S(t)}]. \tag{30}$$

Here $S(t)$ is a variable which indicates the position of the center of the kink. The effective Lagrangian describes the dynamics of this variable. In particular the kinetic energy is the following:

$$T = 2abJ(S)\dot{S}^2 \approx 4ab\dot{S}^2. \tag{31}$$

In the above formula we used approximation $J(S) = \tanh(l-S)+\tanh(S) \approx 2$. The validity of this approximation in the interval $0<S<l$ follows from Fig. 2. The potential energy can be easily evaluated if we assume that the curvature radius of the junction is larger than its width; i.e., for $K(s) \ll 1/a$,

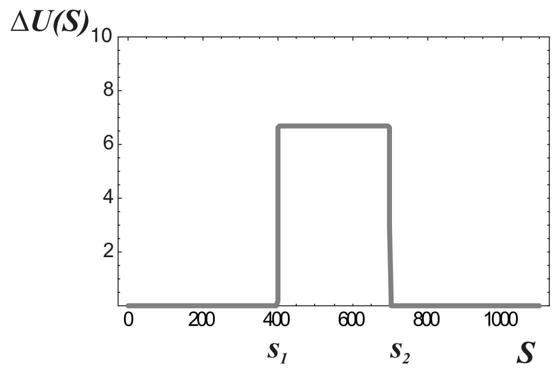

FIG. 3. The potential barrier in the first example is constant in the region $s \in [s_1, s_2]$. This picture is valid whenever $0 \ll s_1 < s_2 \ll l$. The parameters in the plot are the following: $a=20$, $R=200$, $b=100$, $l=1000$, $s_2=700$, and $s_1=400$.

$$U \approx 8ab + \frac{ba^3}{6}\int_0^l ds \frac{K(s)^2}{\cosh^2(s-S)} \equiv 8ab + \Delta U. \quad (32)$$

This potential consists of some constant and the energy barrier ΔU that fixes the dynamics of the kink. We consider two examples of the curved junctions.

Example 1. In the first example the junction consists of two straight segments $K(s)=0$ for $s \in [0,s_1] \cup [s_2,l]$ connected by the arc of the circle of radius R, i.e., $K(s)=1/R$ =const for $s \in [s_1,s_2]$. The energy barrier in this case has extremely simple form (see Fig. 3),

$$\Delta U \approx \frac{ba^3}{6R^2}\frac{\sinh(s_2-s_1)}{\cosh(S-s_1)\cosh(s_2-S)}. \quad (33)$$

We have in mind that we used approximation which is valid for $R \gg a$. The result obtained in this example has extensive description in [7], where it was treated analytically and numerically as well.

Example 2. In a generic situation curvature in a bent section is not constant. In order to demonstrate the effect of the changeable curvature on the shape of the barrier, we will study a different example. Let us consider two straight segments $K(s)=0$ for $s \in [0,s_1] \cup [s_2,l]$ connected by the arc with linearly growing curvature, i.e., $K(s)=s/R$=const, for $s \in [s_1,s_2]$. The energy barrier in this case has a linear slope also (see Fig. 4),

$$\Delta U \approx \frac{ba^3}{6R^2}\left[\ln\left(\frac{\cosh(S-s_1)}{\cosh(S-s_2)}\right) + s_1 \tanh(S-s_1) - s_2 \tanh(S-s_2)\right]. \quad (34)$$

This example suggests close correspondence between the curvature of the junction and the shape of the potential barrier.

B. Case 2

In the second case, due to explicit dependence of the kink profile on the normal variable ρ [i.e., $\phi_K=\phi_K(t,s,\rho)$], the system cannot be treated as one dimensional. Although the system is two dimensional, the formalism proposed in this paper still enables a description of some sort of this type of junction. Those two-dimensional junctions have to be flat in the binormal direction. Now, due to used formalism, we are limited to description of the junctions flat in the direction of the normal variable ρ. In the considered case $\partial_u \phi_K=0$ and $\partial_\rho \phi_K \neq 0$, therefore Lagrangian density (24) takes the following form:

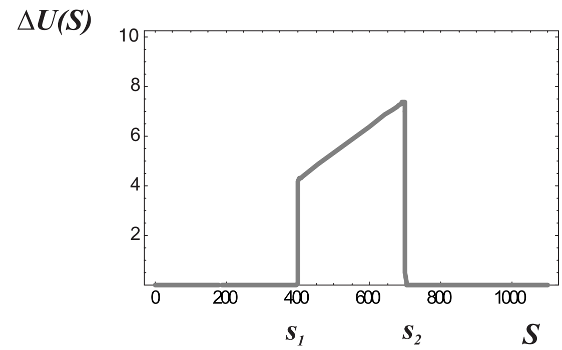

FIG. 4. The potential barrier in the second example, similar to curvature, grows linearly. The parameters (l,a,b,s_1,s_2) of this plot are identical with the parameters in Fig. 3 and $R=5000$.

$$\mathcal{L} = \frac{1}{2}(\partial_t\phi_K)^2 - \frac{1}{2}(\partial_\rho\phi_K)^2 - \frac{\omega^2}{2G}u^2(\partial_\rho\phi_K)^2 - \frac{\omega}{G}u(\partial_\rho\phi_K)(\partial_s\phi_K) - \frac{1}{2G}(\partial_s\phi_K)^2 - V(\phi_K). \quad (35)$$

An example of this kind of kink ansatz is motivated by [8] and has the form

$$\phi_K = 4\arctan[e^{s-S(t)+\rho}], \quad (36)$$

where $S(t)$ specifies the kink position. Now we calculate the partial derivative of this adiabatic ansatz,

$$\dot{\phi}_K^2 = \frac{4}{\cosh^2(s-S+\rho)}\dot{S}^2, \quad (37)$$

and obtain the effective kinetic energy

$$T = 2aI(S)\dot{S}^2, \quad (38)$$

where the function $I(S)$,

$$I(S) = \ln\left[\frac{(e^{2S}+e^{b+2l})(e^{b+2S}+1)}{(e^{2S}+e^b)(e^{b+2S}+e^{2l})}\right], \quad (39)$$

has the simple profile presented in Fig. 5. The function $I(S)$ is constant in the central part of the junction and linearly depends on the S variable at the ends,

$$I(S) \approx \begin{cases} 2S+b & \text{for } 0 \le S \le b/2 \\ 2b & \text{for } b/2 \le S \le l-b/2 \\ 2l-2S+b & \text{for } l-b/2 \le S \le l. \end{cases} \quad (40)$$

The effective potential energy, if the radius of curvature is much larger than the thickness of the junction (i.e., if $K \gg 1/a$), is the following:

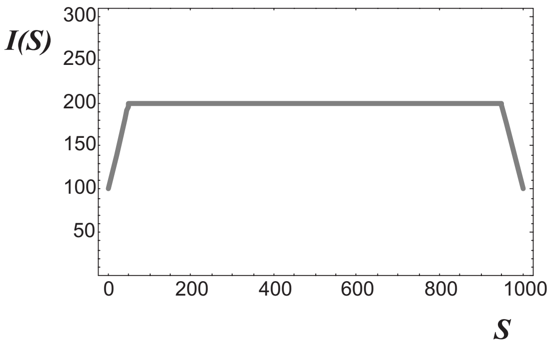

FIG. 5. The function $I(S)$ is constant almost in the whole interval $[0,l]$. The dependence of the function $I(S)$ on the S variable is linear only at the ends of the junction. In the plot we have chosen $b=100$ and $l=1000$.

$$U \approx 6aI(S) + \frac{a^3}{6}\int_0^l ds \int_{-b/2}^{+b/2} d\rho \frac{\left(1+\frac{\omega(s)}{K(s)}\right)K^2(s)}{\cosh^2(s-S+\rho)}. \quad (41)$$

Particular example of this type of the junction consists of the straight segments $[0,s_1]\cup[s_2,l]$, where $K(s)=\omega(s)=0$, connected by the curved region $[s_1,s_2]$ which has a form of a helix with constant curvature $(K=1/R)$ and constant torsions (18) and (19). The potential energy of the kink in this junction has the form

$$U \approx 6aI(S) + \frac{a^3}{6R^2}(1+\omega R)H(S), \quad (42)$$

where $H(S)$ is the following function of the adiabatic variable (see Fig. 6):

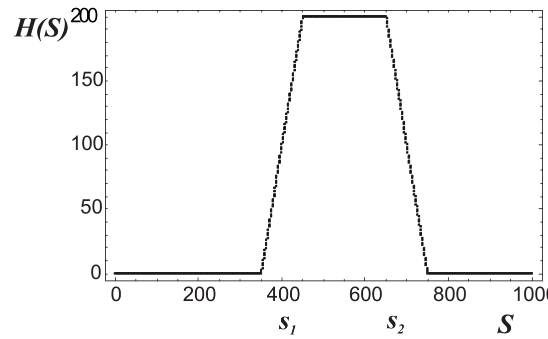

FIG. 6. The function $H(S)$ defines the potential barrier in the third example. The shape of this function is generic whenever $0 \ll s_1 < s_2 \ll l$. The choice of parameters (b,l,s_1,s_2) is identical to the choice in Fig. 3.

$$H(S) = \ln\left[\frac{(e^{2S}+e^{b+2s_2})(e^{b+2S}+e^{2s_1})}{(e^{2S}+e^{b+2s_1})(e^{b+2S}+e^{2s_2})}\right]. \quad (43)$$

If we skip unimportant constant $(-12ab)$, then the effective Lagrangian, in the central part of the junction (i.e., for $l-b/2 \geq S \geq l$), is the following:

$$L \approx 4ab\dot{S}^2 - \frac{a^3}{6R^2}(1+\omega R)H(S) \equiv 4ab\dot{S}^2 - \Delta U, \quad (44)$$

where we used the simplified form of the function $I(S)$ [Eq. (40)]. The dynamics of the kink in this region is determined solely by the potential barrier ΔU. On the other hand at the ends the junction is straight so $\Delta U=0$. In these regions the function $I(S)$ changes linearly [Eq. (40)], and therefore the effective Lagrangian simplifies to the form

$$L \approx \begin{cases} 2a(2S+b)\dot{S}^2 - 6a(2S+b) & \text{for } 0 \leq S \leq b/2 \\ 2a(2l-2S+b)\dot{S}^2 - 6a(2l-2S+b) & \text{for } l-b/2 \leq S \leq l. \end{cases}$$

If we replace the variable S by the variable $q=(2S+b)^{3/2}/3$ (for $0 \leq S \leq b/2$), and variable S by the variable $q=(2l-2S+b)^{3/2}/3$ (for $l-b/2 \leq S \leq l$), then at both ends the Lagrangian simplifies to the form

$$L \approx 4a\left[\frac{1}{2}\dot{q}^2 - \frac{3}{2}(2q)^{2/3}\right]. \quad (45)$$

This form of Lagrangian (45) suggests the existence of some energy barrier connected with inserting the kink into the junction. The meaning of this barrier is clearly visible in Fig. 7. In order to insert kink (36) into the junction initially, we have to increase the length of the kink. This observation also suggests the simple way of confining the kink in some regions of the junction. According to this observation the appropriate potential distribution can be arranged even in the straight junction and even for the simple kink (30) if we change the width $[-b/2,b/2]$ along the junction (i.e., along the s direction). In the junction deformed in this way, the kink can be confined even in the narrow regions of the Josephson junction (see Fig. 8).

IV. REMARKS

In the present paper, we presented a formalism that allows for the description of the motion of the kink in an arbitrarily deformed Josephson junction. The paper is based on the collective coordinate method and allows one to find an effective Lagrangian that describes the dynamics of the adiabatic variable indicating the position of the kink. In the present work, we generalize the results of [7], which presents the junction as a curve embedded in a plane. The generalization proposed

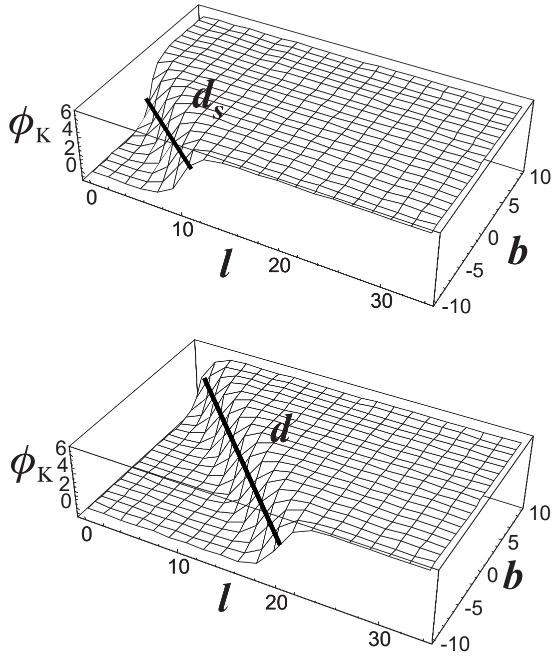

FIG. 7. At the ends of the Josephson junction, the longitude d_s of the profile of the kink is shorter than in the central part of the junction d.

by us allows for an analytical description of the Josephson junction whose shape makes it impossible to reduce the embedding of the junction to two dimensions. This nontrivial embedding in three dimensions is a reason for the occurrence, in the description of the system, of the curvature and torsion as well. We considered three examples of curved junctions. In the first two examples we considered one-dimensional (long Josephson junction) and quasi-one-dimensional junctions. The effective potential in these examples and for any other flat deformations of the junction follows from formula (32). In the first example, we recover the results of [7]. In the second example, we showed a direct connection between the shape of the barrier in the effective model and the curvature $K(s)$ of the junction. The proposed formalism can also be used in order to describe the motion of the kink in some sort of the two-dimensional junctions (junctions flat in the binormal direction). The effective potential in the case of the two-dimensional junction embedded in three dimensions is given in formula (41). In the last example, we show a situation where the embedding in three dimensions is nontrivial and therefore the presence of the torsion in the effective description is inevitable. Studying this generically two-dimensional junction embedded in three dimensions, we also noticed the possibility of forming an arbitrary potential even in a straight junction for a quasi-one-dimensional kink. A simple way of preparing a junction with appropriate potential is by changing the width of the junction in the normal direction (in the direction of the ρ variable). If the width of the junction changes, for example, if it grooves, then we have to provide some energy connected with increasing the length of the kink profile (Fig. 8). The possibility of creating a potential hole (for example, made of two barriers) could find future applications in electronic devices in order to store binary data. These applications are related to the possibility of confining the fluxons in potential holes. The process of trapping fluxons in potential holes may be controlled by introducing the external magnetic field (in order to push the fluxon into a potential hole) and small damping to the system (in order to decrease the kinetic energy of the kink).

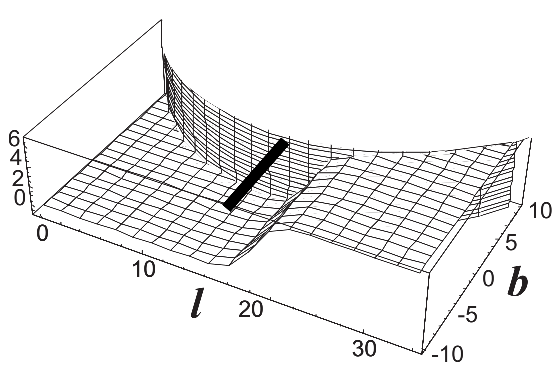

FIG. 8. The Josephson junction with changeable width. The minimum of the potential hole is located at the position of the central bar.

The effective description of the kink motion, proposed in this paper, is based only on position variable. We would like to underline that at this point we rely on the numerical test made for the curved system described in the first example. Extensive numerical studies made in [7] show that position variable suffices for good description of the kink motion for nonrelativistic speeds. We understand that the description of the kink motion for relativistic speeds needs some modifications (in fact [9] suggests the need of some kind of the width variable), but we consider it as a good idea for future work.

Finally, let us think about possible dependence of the kink amplitude on curvature of the junction. We know that the field variable ϕ measures the phase difference

$$\phi(t,\vec{x}) = \varphi_2(t,\vec{x}) - \varphi_1(t,\vec{x})$$

of the many-particle functions that describe both superconducting electrodes

$$\psi_1(t,\vec{x}) = |\psi_0| e^{\varphi_1(t,\vec{x})}, \quad \psi_2(t,\vec{x}) = |\psi_0| e^{\varphi_1(t,\vec{x})},$$

where $|\psi_0|$ is a wave-function amplitude. It is assumed that in the bulk of the superconductor this amplitude is constant and a phase is a dominating degree of freedom. It is well-established fact that the phase of the wave function is sensitive to the topology of the space (due to requirement of the uniqueness). On the other hand if the topology is trivial (this is the case of the considered system), one could ask whether the phase depends on the curvature of the superconducting electrode. The reliable answer can be achieved in the framework of the microscopic theory (BCS for conventional superconductors). In this theory we can ask about possible dependence of the effective many-particle function on the curvature of the boundaries of the superconducting material.

To make things simpler let us consider a simple example, i.e., one-dimensional quantum mechanical (QM) system. We would like to compare a wave function defined on the section of the straight line

(1-d flat system) with a wave function defined on a segment of an arc of the circle (1-d curved system). We are interested in the form of the Laplacian in both systems as it is the only term in the stationary Schrödinger equation that may depend on the geometry of the system. In case of the circle we write down the Laplacian in polar coordinates and then fix the radial coordinate. Next we change the angle variable to length parameter and obtain the identical form of the Laplacian as in the case of the section on the straight line. As a result we obtain the same analytical form of the functions parametrized by the same length parameter. This instance suggests the lack of dependence of the phase factor on the curvature and therefore also the lack of the dependence of the kink amplitude on a curvature of the junction.

[1] B. D. Josephson, Phys. Lett. **1**, 251 (1962).
[2] P. W. Anderson and J. M. Rowell, Phys. Rev. Lett. **10**, 230 (1963).
[3] B. D. Josephson, Adv. Phys. **14**, 419 (1965); A. Barone and G. Paterno, *Physics and Applications of the Josephson Effect* (Wiley, New York, 1982).
[4] O. Babelon, D. Bernard, and M. Talon, *Introduction to Classical Integrable Systems* (Cambridge University Press, Cambridge, England, 2003).
[5] M. J. Ablowitz and P. A. Clarkson, *Solitons, Nonlinear Evolution Equations and Inverse Scattering* (Cambridge University Press, Cambridge, England, 1999); L. A. Ferreira, B. Piette, and W. J. Zakrzewski, Phys. Rev. E **77**, 036613 (2008); J. Phys.: Conf. Ser. **128**, 012027 (2008).
[6] B. A. Malomed, Physica D **52**, 157 (1991); J. Zagrodziński, Phys. Lett. **72**, 284 (1979); G. Leibbrandt, Phys. Rev. Lett. **41**, 435 (1978); K. K. Kobayashi and M. Izutsu, J. Phys. Soc. Jpn. **41**, 1091 (1976).
[7] C. Gorria, Yu. B. Gaididei, M. P. Soerensen, P. L. Christiansen, and J. G. Caputo, Phys. Rev. B **69**, 134506 (2004).
[8] D. Ouroushev, N. Martinov, and A. Grigorov, J. Phys. A **24**, L527 (1991).
[9] M. J. Rice, Phys. Rev. B **28**, 3587 (1983); J. C. Fernandez, M. J. Goupil, O. Legrand, and G. Reinisch, *ibid.* **34**, 6207 (1986); E. Turlot, D. Esteve, C. Urbina, M. Devoret, R. Grauer, J. C. Fernandez, H. Politano, and G. Reinisch, *ibid.* **42**, 8418 (1990).